ISRAEL JOURNAL OF MATHEMATICS, Vol. 38, Nos. 1-2, 1981

DUAL BANACH LATTICES AND BANACH LATTICES WITH THE RADON–NIKODYM PROPERTY

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ABSTRACT

We construct a separable dual Banach lattice E such that no non-trivial order interval of its dual is weakly compact. Hence E has the Radon–Nikodym property without being in some sense a dual in a natural way.

I. Introduction

It has been an open problem for a long time whether every separable Banach space with the Radon–Nikodym Property (RNP — see [2]) is a subspace of a separable dual Banach space. It is known now that the answer is negative [1], [4]. These two examples are very different, but both of them are far from being a Banach lattice. In fact, it does not seem to be even known today if a separable Banach lattice with RNP is a dual (and this question will not be answered here). An interesting idea in this direction is due to H. P. Lotz [3], along the following lines. Let E be a separable Banach lattice satisfying RNP. Let F be the set of x in the dual E^* of E such that $[0, |x|]$ is weakly compact. Then F is a Banach lattice. Lotz shows that if F is big enough, i.e. $\sigma(E, F)$ is Hausdorff, then $E = F^*$. Hence F is a natural candidate as a predual of E. The purpose of this paper is to describe an example (whose existence is claimed in [5]) of a separable Banach lattice E (which is a dual) such that $F = \{0\}$. Hence, if E is a separable Banach lattice satisfying RNP, there does not seem to exist a natural candidate for a predual. We feel that this means both that, in general, E is not likely to be a dual, and that the problem is not likely to be easy.

[†] The final draft of this paper was written while the author held a grant from NATO to visit the Ohio State University.

Received January 30, 1980

II. The example

THEOREM. *There exists a separable Banach lattice E with the following two properties*:

(a) *E is a dual* (*and hence has* RNP);

(b) *if* $x \in E^*$, $x \neq 0$, *then there exists a sublattice isomorphic to* l^∞, *which unit ball is contained in* $[-|x|,|x|]$ (*and hence* $[0,|x|]$ *is not weakly compact*).

PROOF. *1st Step. Construction.* Let $K = \{0,1\}^{\mathbf{N}}$ be the Cantor set, and λ its canonical measure. Let $(A_k^n)_{n \geqq 0, k \geqq 0}$ be a family of clopen sets of K, which are independent for λ, and such that for all n and k, $\lambda(A_k^n) = 2^{-n}$. (The existence of such a family is obvious if one remembers that K is isomorphic to $\{0,1\}^{\mathbf{N}\times\mathbf{N}}$.) Let $y_{n,k} = 2^n \chi_{A_k^n}$. Hence $\|y_{n,k}\|_1 = 1$. Let $L_1 = L_1(\lambda)$, $L_2 = L_2(\lambda)$. Let

$$W = \left\{ z \in L_1, \exists x \in L_2^+, \exists a_{n,k} \geqq 0, \sum_{n,k} a_{n,k} < +\infty, |z| \leqq x + \sum_{n,k} a_{n,k} y_{n,k} \right\}.$$

For $z \in E$, let $\|z\|_E = \operatorname{Inf}\{\|x\|_2 + \Sigma_{n,k}\, a_{n,k}\}$, where the inf is taken over the families $x \in L_2^+$, $(a_{n,k})$ with $|z| \leqq x + \Sigma_{n,k}\, a_{n,k} y_{n,k}$. It is standard to show that E is a Banach lattice.

Let us first show that $\mathscr{C}(K)$ is dense in E. Since $\|\ \|_\infty$ is stronger than $\|\ \|_E$ on $\mathscr{C}(K)$, this will show that E is separable. Let $z \in E$. We have $|z| \leqq x + \Sigma_{n,k} a_{n,k} y_{n,k}$, where $x \in L_2$ and $\Sigma_{n,k} a_{n,k} < +\infty$. Hence, $z = z_1 + z_2$, where $|z_1| \leqq x$, $z_2 \leqq \Sigma_{n,k} a_{n,k} y_{n,k}$. Let $\varepsilon > 0$. Since $|z_1| \leqq x$, $z_1 \in L_2$, there exists $t_1 \in \mathscr{C}(K)$ with $\|z_1 - t_1\|_2 \leqq \varepsilon/3$. Let I be a finite subset of $\mathbf{N}\times\mathbf{N}$ such that $\Sigma_{(n,k)\notin I} a_{n,k} \leqq \varepsilon/3$. We have $z_2 = z_3 + z_4$, where $|z_3| \leqq \Sigma_{n,k \in I} a_{n,k} y_{n,k}$, $|z_4| \leqq \Sigma_{(n,k)\notin I} a_{n,k} y_{n,k}$. Since z_3 is bounded, there exist $t_3 \in \mathscr{C}(K)$ with $\|z_3 - t_3\|_2 \leqq \varepsilon/3$. Now

$$|z - t_1 - t_3| \leqq |z_1 - t_1| + |z_3 - t_3| + |z_4|$$

where $\| |z_1 - t_1| + |z_3 - t_3| \|_2 \leqq 2\varepsilon/3$, $|z_4| \leqq \Sigma_{(n,k)\notin I}\, a_{n,k} y_{n,k}$ and $\Sigma_{(n,k)\notin I}\, a_{n,k} \leqq \varepsilon/3$, so $\|z - t_1 - t_3\|_E \leqq \varepsilon$.

Next, we have $L_2 \hookrightarrow E \hookrightarrow L_1$, where each of the injections is positive, of norm $\leqq 1$ (for the second one, this follows from $\|x\|_1 \leqq \|x\|_2$, $\|y_{n,k}\|_1 = 1$). Hence, for each n, k, $1 = \|y_{n,k}\|_1 \leqq \|y_{n,k}\|_E \leqq 1$, so $\|y_{n,k}\|_E = 1$. Since $L_2 \hookrightarrow E$, it is clear that E' can be identified with those $t \in L_2$ for which there exist a constant M such that $\int |t| y_{n,k} \leqq M$ for all n, k. In particular, $\mathscr{C}(K) \hookrightarrow E^*$, and since the norm of E is stronger than the norm of L_1, this injection is continuous when $\mathscr{C}(K)$ is provided with the L_∞ norm. This shows that the image of $\mathscr{C}(K)$ in E^* is separable.

2nd Step. We are going to show that E is the dual of the closure of the image of $\mathscr{C}(K)$ in E^*. Since this image is separable, it is enough to show that each sequence (z^p) in the unit ball of E has a subsequence which converges for $\sigma(E,\mathscr{C}(K))$. (Hence the unit ball of E will be $\sigma(E,\mathscr{C}(K))$-compact.) For each p, one can write $z^p = z_1^p + z_2^p$, where $z_1^p \in L_2$, $|z_2^p| \leqq \Sigma_{n,k} a_{n,k}^p y_{n,k}$ and $\|z_1^p\|_2 + \Sigma_{n,k} a_{n,k}^p \leqq 1 + 2^{-p}$. By taking a subsequence, one can assume that $\|z_1^p\|$ converges, to α say, that z_1^p converges weakly in L_2 to z_1 (with $\|z_1\| \leqq \alpha$) and that for each n,k $(a_{n,k}^p)_p$ converges to $a_{n,k}$. One can by taking another subsequence assume that $a_{n,k}^p \leqq a_{n,k} + 2^{-n-k-p}$ for $n,k \leqq p$. For each p, one can write $z_2^p = \Sigma_{(n,k)\in I_p} z_{n,k}^p + z_3^p$ where $I_p = [0,p]\times[0,p]$, $|z_{n,k}^p| \leqq a_{n,k}^p y_{n,k}$, $|z_3^p| \leqq \Sigma_{(n,k)\notin I_p} a_{n,k}^p y_{n,k}$. It is also possible to assume that $z_{n,k}^p \to z_{n,k}$ say in the $\sigma(L_1,\mathscr{C}(K))$ topology. Of course, $|z_{n,k}| \leqq a_{n,k} y_{n,k}$.

We show now that $\Sigma_{(n,k)\in I_p} z_{n,k}^p \to \Sigma_{n,k} z_{n,k}$ for $\sigma(E,\mathscr{C}(K))$. For $f \in \mathscr{C}(K)$, we have for any p_0

$$\left| \sum_{(n,k)\in I_p} \int f z_{n,k}^p - \sum_{n,k} \int f z_{n,k} \right| \leqq \|f\|_\infty \Bigg(\sum_{(n,k)\in I_{p_0}} \int |z_{n,k}^p - z_{n,k}| + \sum_{\substack{(n,k)\notin I_{p_0}\\ (n,k)\in I_p}} a_{n,k}^p + \sum_{(n,k)\notin I_{p_0}} a_{n,k} \Bigg).$$

Since $a_{n,k}^p \leqq a_{n,k} + 2^{-n-k-p}$ for $(n,k)\in I_p$, we have

$$\sum_{\substack{(n,k)\notin I_{p_0}\\ (n,k)\in I_p}} a_{n,k}^p \leqq 2^{-p_0} + \sum_{(n,k)\notin I_{p_0}} a_{n,k}.$$

So, if $\varepsilon > 0$ and we let p_0 be large enough such that $2^{-p_0} + \Sigma_{(n,k)\notin I_{p_0}} a_{n,k} \leqq \varepsilon$, we have

$$\limsup_p \left| \int f \sum_{(n,k)\in I_p} z_{n,k}^p - \int f \sum_{n,k} z_{n,k} \right| \leqq 2\varepsilon$$

which proves our assertion, since ε is arbitrary.

We still have to study the behavior of z_3^p. Let $\beta_p = \Sigma_{(n,k)\notin I_p} a_{n,k}^p$. We can suppose $\beta = \lim \beta_p$ exist. We can also suppose that z_3^p converges in $\mathscr{C}(K)^*$ to a measure μ. We have $\|\mu\| \leqq \beta$. We are going to prove that $d\mu = h\,d\lambda$, where $\|h\| \leqq \beta$. It is enough to show that $\mu(X) \leqq \beta\lambda(X)$ for each clopen set X of L. Let p_0 be large enough such that for $(n,k)\notin I_{p_0}$, for $A_{n,k}$ independent of the coordinates of which X depends. It is enough to show that $|\int_X z_3^p| \leqq \beta_p \lambda(X)$ for $p \geqq p_0$. But it is clear that z_3^p is limit in L^1 of elements of the type $\Sigma_{(n,k)\notin I_p,(n,k)\in I_q} t_{n,k}$ where $|t_{n,k}| \leqq a_{n,k}^p y_{n,k}$. But we have

$$\int_X y_{n,k} = 2^n \lambda(X \cap A_{n,k}) = \lambda(X)$$

so the claim about μ follows. Hence $z_3^p \to h$ for $\sigma(E, \mathscr{C}(K))$. Of course, $z_1^p \to z_1$ for $\sigma(E, \mathscr{C}(K))$, since this topology is coarser on L_2 than $\sigma(L_2, L_2)$.

So $z \to z_1 + \Sigma_{n,k} z_{n,k} + h$ for $\sigma(E, \mathscr{C}(K))$, and norm of the limit is $\leqq \alpha + \Sigma_{n,k} a_{n,k} + \beta$, and it is clear that this is $\leqq 1$, since $\limsup_p \Sigma_{n,k} a_{n,k}^p \leqq 1 - \alpha$.

Hence E is the dual of the image of $\mathscr{C}(K)$ in E^*.

3rd Step. Let $x \in E'$, $x > 0$. We have shown that there exist a measurable set $B \subset K$, and $\alpha > 0$ with $x \geqq \alpha \chi_B$. In order to prove (b), we can hence assume that $x = \chi_B$. Let $\beta = \lambda(B)/3$.

For all n, $(1 - 2^{-n})^{2^n} \geqq 1/2e \geqq 1/6$. Let p be an integer such that $1/6^p \leqq \beta$. For all n, we have $\lambda(\bigcup_{k \leqq p2^n} A_{n,k}) \geqq 1 - \beta$. We are going to construct by induction a sequence (k_l) of integers, such that if l_0 is fixed such that $2^{-l_0} \leqq \beta$, the following condition is satisfied for all l, where $A_l = A_{l_0+l+1,k_l}$:

$$\lambda\left(A_l \cap \left(B \setminus \bigcup_{l'<l} A_{l'}\right)\right) \geqq \beta p^{-1} \lambda(A_l). \tag{1}$$

The first step being the same as the general step let us assume the construction has been done for $l' < l$. We have $\lambda(\bigcup_{l'<l} A_{l'}) \leqq \Sigma_{l \geqq l_0} 2^{-l_0-l-1} \leqq 2^{-l_0} \leqq \beta$ so if we set $C = B \setminus \bigcup_{l'<l} A_{l'}$, we have $\lambda(C) \geqq 2\beta$. We claim there exist $k \leqq p2^{l_0+l+1}$ with $\lambda(A_{l_0+l+1,k} \cap C) \geqq \beta p^{-1} \lambda(A_{l_0+l+1,k})$. For otherwise, we would have

$$\beta \leqq \lambda\left(\bigcup_{k \leqq p2^{l_0+l+1}} A_{l_0+l+1,k} \cap C\right) \leqq \sum_{k \leqq p2^{l_0+l+1}} \lambda(A_{l_0+l+1,k} \cap C)$$

$$< p2^{l_0+l+1} \beta p^{-1} 2^{-(l_0+l+1)} = \beta,$$

a contradiction. So if we take $k_l = k$, this concludes the construction.

Now, for each l, let $C_l = A_l \cap (B \setminus \bigcup_{l'<l} A_{l'})$. These sets are disjoint, and from (1) we get, if $y_l = y_{l+l_0+1,k_l}$,

$$\|\chi_{C_l}\|_{E'} \geqq \int y_l \chi_{C_l} \geqq \beta p^{-1}.$$

It is thus clear that the map $(t_n) \to \Sigma_l t_l \chi_{C_l}$ is an isomorphism of l^∞ and a sublattice of E^*, whose unit ball is contained in $[-|x|, |x|]$. The proof is finished.

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